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Hénon-like maps and renormalisation

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Publisher's PDF, also known as Version of record

Publication date:

2008

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Hazard, P. E. (2008). *Hénon-like maps and renormalisation*. s.n.

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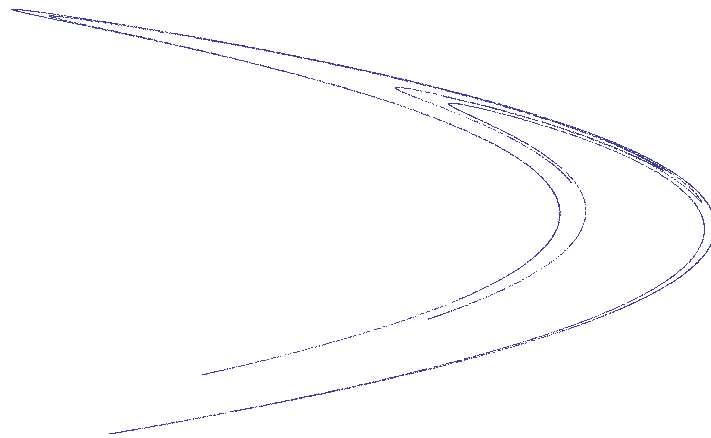
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Hénon-like Maps and Renormalisation



P E Hazard

RIJKSUNIVERSITEIT GRONINGEN

Hénon-like Maps and Renormalisation

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
maandag 8 december 2008
om 11.00 uur

door

Peter Edward Hazard

geboren op 21 februari 1982
te Middlesbrough, Engeland

Promotores: Prof. dr. H.W. Broer
dr. ir. M. Martens

Beoordelingscommissie: Prof. dr. M. Benedicks
Prof. dr. A. de Carvalho
Prof. dr. S. van Strien

Agreement of Joint Program

The following is a dissertation submitted in partial fulfillment of the requirements for the degree Doctor of Philosophy in Mathematics awarded jointly by Rijksuniversiteit Groningen, The Netherlands and Stony Brook University, USA. It has been agreed that neither institution shall award a full doctorate. It has been agreed by both institutions that the following are to be the advisors and reading committee.

Advisors: Prof. H.W. Broer, Rijksuniversiteit Groningen
 Assoc. Prof. M. Martens, Stony Brook University

Reading Committee: Prof. M. Benedicks, KTH Stockholm, Sweden
 Prof. S. van Strien, University of Warwick, UK
 Ass. Prof. A. de Carvalho, USP Sao Paulo, Brazil
 Assoc. Prof. S. Sutherland, Stony Brook University
 Dr. J. Kahn, Stony Brook University
 Dr. R. Roeder, Stony Brook University

Chair of the Defense: Prof. W.C. Nieuwpoort, Rijksuniversiteit Groningen

Both institutions agree that the defense of the above degree will take place on Monday 8th December 2008 at 11:00am at the Academieggebouw, Rijksuniversiteit, Groningen, The Netherlands and that, if successful, the degree Doctor of Philosophy in Mathematics will be awarded jointly by Rector Magnificus, dr. F. Zwarts, RuG, Groningen, and Prof. L. Martin, Graduate School Dean, Stony Brook University.

I hereby agree that I shall always describe the diplomas from Stony Brook University and Rijksuniversiteit Groningen as representing the same doctorate.

Peter Edward Hazard

Dedicated to my Mother, Father

Sister and Brother

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Acknowledgements

I would like to take this opportunity to express my gratitude to everyone who helped me during the completion of this work, and my graduate school career in general. For any omissions I apologise. Most importantly, I would like to thank my advisor, Marco Martens. Words are not enough to express my appreciation of his help, guidance and encouragement throughout the last four years.

I also thank Sarma Chandramouli. We joined RuG at almost exactly the same time and came to Stony Brook together with our advisor. From the beginning he has been a good friend and a helpful colleague. I thank him for many fruitful renormalisation discussions and for helping with many of the pictures here.

The remainder is separated into two parts, corresponding to my time at RuG and Stony Brook respectively. Firstly, thanks to Björn Winckler, Olga Lukina, Stephen Meagher and Lenny Taelmann for the time I spent with them at RuG. The discussions we had, both mathematical and otherwise, were very enjoyable and enlightening. I also give special thanks to Colin Hazard and his family for treating me like a son during my stay in the Netherlands. Without them I would not understand or appreciate the Dutch way of life nearly as much as I do now. In Stony Brook, I would like to thank Myong-hi Kim and Nadia Kennedy for helping me settle when I first arrived. Similarly thanks goes to Allison Haigney, Andrew Bulawa, Carlos Martinez-Torteya and Gedas Gasparavicius for their help, patience and understanding during the chaotic parts of my last few months here.

A general mention goes to the Stony Brook Guitar Dectet for providing me with, maybe too many, distractions and everyone on the second floor of the Math Tower for help and inspiration, both mathematical or otherwise. I have learnt a lot from everyone there. I will not name everyone for there are too many of them, but they know who they are.

I now like to thank the members of my reading committee. To Sebastian van Strien, for introducing me to the field of Dynamical Systems as an undergraduate. Without his inspiration, encouragement and guidance I would not be here now. To Michael Benedicks for sharing his insights into the dynamics of Hénon maps and on renormalisation and the various useful and interesting mathematical discussions we had during his stay at Stony Brook in the Spring of 2008. Finally, to André de Carvalho for introducing me to pruning fronts and

useful discussions on the topological definition of Hénon renormalisation during my stay at USP Sao Paulo during the Summer of 2008.

On a technical note, thanks goes to Han Peters for help with the Samenvatting and Pedro Solorzano and Ki Song for proof-reading parts of this work. Their comments and suggestions were crucial to the writing process. I would also like to thank all the staff at both RuG and Stony Brook for their help in making this dissertation possible, in particular Donna McWilliams in Stony Brook and Yvonne van der Weerd and Elizabeth Woldringh in Groningen. Also to Henk Broer for all the arrangements he made on my behalf over the last six months.

Finally I thank my family for being my family, for their patience and understanding, and giving me all the support I needed throughout this adventure. I hope I have made them proud.